

An open FRW model in Loop Quantum Cosmology

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Abstract

Open FRW model in Loop Quantum Cosmology is under consideration. The left and right invariant vector fields and holonomies along them are studied. It is shown that in the hyperbolic geometry of $k = -1$ it is possible to construct a suitable loop which provides us with quantum scalar constraint originally introduced by Vandersloot [19]. The quantum scalar constraint operator with negative cosmological constant is proved to be essentially self-adjoint.

1 Introduction

Loop Quantum Cosmology (LQC) is a novel approach to quantum theory of cosmology [8, 9, 10]. The Loop Quantum Gravity (LQG) [18, 17, 2] inspired quantization of the symmetry reduced models which are the test field for the full theory. It also provides some very interesting results like quantum geometry effects and absence of singularity [11]. During last years there has been a progress in the area of LQC [12, 11, 1, 4, 13]. Although the part of isotropic ($k = 0$ [6] and $k = 1$ [12, 15, 7]) and homogeneous sector of quantum theory is well understood there still is a problem with Bianchi class B models such as open Friedmann Robertson Walker (FRW) model (so called $k = -1$)¹. One of the reasons for that is the well known problem concerning the Hamiltonian formulation of class B models. However, one can derive that the isotropic $k = -1$ model in terms of real Ashtekar variables has correct Hamiltonian formulation (see [19] and references therein). The second problem comes from the geometric difficulty. It is not clear how to introduce a loop suitable for the quantization purposes. Although there has been a recent progress in the FRW hyperbolic model [19] a potential gap arose. The object which was quantized in [19] was the holonomy along an open curve. However, such a

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¹Strictly speaking the isotropic $k = -1$ model is derived from anisotropic Bianchi V

holonomy should not be considered as the components of the curvature. Moreover, the holonomies considered in [19] were used with respect to the $\gamma K = A - \Gamma$ variables rather than to the A connection (there is nothing wrong with it logically, but it makes the relation to the full theory obscure). However, quantum theory described in [19] does not suffer from singularities when gravity is coupled to homogeneous massless scalar field and has a correct classical limit as well. In that sense the theory of quantum $k = -1$ model is correct and provides new quantum gravitational effects. One can conclude that [19] is a major candidate to replace the old classical $k = -1$ model by the new quantum one. We show in the present paper how this model can become conceptually closer to the full (LQG) theory by introducing a suitable loop, which leads to the same scalar constraint as in [19]. Using a new loop in quantum theory has one more advantage. The implementation of “improved dynamics” introduced by Ashtekar, Pawłowski and Singh in [6] is direct and natural what is missing in [19]. However, our results are valid again only for the γK holonomies.

This paper is organized as follows. In section 2 we briefly discuss the classical hyperbolic geometry of $k = -1$ model as well as its Hamiltonian formulation. In section 3 the quantum scalar constraint is constructed in detail by introduction of a suitable loop. Section 4 describes properties of the scalar constraint operator with negative cosmological constant.

2 Classical Theory

2.1 FRW models

The well known isotropic and homogeneous sector of the General Relativity can be considered as three so called Friedmann-Robertson-Walker models, where the metric tensor has a form

$$g = -N^2(t)dt^2 + a(t)^2[(1 - kr)^{-1}dr^2 + r^2d\Omega^2]. \quad (2.1)$$

Because of the large number of symmetries there is only one gravitational degree of freedom, the scale factor $a(t)$ which is a function of arbitrarily chosen time coordinate t . $N(t)$ is referred to as a laps function and does not enter the equations of motion as a dynamical variable. k is a number which can take only three values. Each of the values of k corresponds to different intrinsic curvature of spatial manifold Σ . Spatially flat, closed and open universes are described by $k = 0, +1, -1$ respectively. Einstein equations for (2.1) describe the dynamics of the scale factor

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{\text{matter}}. \quad (2.2)$$

This equation describes evolution of the universe filled by matter density ρ_{matter} .

2.2 The $k = -1$ geometry

It is well known [20] that spatial part of (2.1) can be written in terms of left invariant one-forms as

$$q = a^2(t) \delta_{ij} \, {}^o\omega_a^i \, {}^o\omega_b^j \, dx^a dx^b := a^2(t) {}^oq_{ab} dx^a dx^b, \quad (2.3)$$

where $i, j = 1, 2, 3$. These one-forms ${}^o\omega_a^i$ satisfy Maurer-Cartan equation

$$\partial_a \, {}^o\omega_b^i = -\frac{1}{2} C_{jk}^i \, {}^o\omega_a^j \, {}^o\omega_b^k, \quad (2.4)$$

where for $k = -1$ structure constants are given by

$$C_{ij}^k = \delta_i^k \delta_{j1} - \delta_j^k \delta_{i1}. \quad (2.5)$$

The same structure constants occur in the algebra of left invariant vector fields² on Σ (it is well known Bianchi class V algebra)

$$[{}^oe_i, {}^oe_j] = C_{ij}^k {}^oe_k. \quad (2.6)$$

Left invariant one-forms and vector fields can be written in some coordinates x^a as

$$\begin{aligned} {}^oe_1 &= \partial_1 & {}^o\omega^1 &= dx^1, \\ {}^oe_2 &= e^{-x^1} \partial_2 & {}^o\omega^2 &= e^{x^1} dx^2, \\ {}^oe_3 &= e^{-x^1} \partial_3 & {}^o\omega^3 &= e^{x^1} dx^3. \end{aligned} \quad (2.7)$$

One can check that equations (2.6) and (2.4) are satisfied by (2.7). Note, that $C_{ij}^i \neq 0$. Such algebras belong to the class B in the Bianchi classification.

2.3 Classical Dynamics

Canonical quantization of full General Relativity as well as symmetry reduced models is based on their Hamiltonian formulation [18, 2]. In terms of Ashtekar variables the full Hamiltonian for GR is a sum of constraints

$$H_{\text{gr}}^{\text{tot}} = \int_{\Sigma} d^3x (N^i G_i + N^a C_a + N h_{\text{sc}}), \quad (2.8)$$

where

$$\begin{aligned} C_a &= E_i^b F_{ab}^i, \\ G_i &= D_a E_i^a := \partial_a E_i^a + \varepsilon_{ij}{}^k A_a^j E_k^a \end{aligned} \quad (2.9)$$

²Left invariant one-forms are dual to left invariant vector fields: ${}^o\omega^i({}^oe_j) = \delta_j^i$, where ${}^o\omega^i = {}^o\omega_a^i dx^a$ and ${}^oe_i = {}^oe_i^a \partial_a$

are called diffeomorphism and Gauss constraints respectively. $F = dA + \frac{1}{2}[A, A]$ is a curvature of Ashtekar connection (2.11). The most complicated scalar constraint has a form

$$H_{\text{gr}} := \int_{\Sigma} d^3x N(x) h_{\text{sc}} = \frac{1}{16\pi G} \int_{\Sigma} d^3x N(x) \left(\frac{E_i^a E_j^b}{\sqrt{|\det E|}} \varepsilon^{ij}{}_k F_{ab}^k - 2(1 + \gamma^2) \frac{E_i^a E_j^b}{\sqrt{|\det E|}} K_a^i K_b^j \right). \quad (2.10)$$

The Ashtekar variables (A, E) are constructed from the triads (see [18, 2] for details) in the following way

$$A_a^i = \Gamma_a^i + \gamma K_a^i \quad E_i^a = \sqrt{|\det q|} e_i^a, \quad (2.11)$$

where $q_{ab} = \delta_{ij} e_a^i e_b^j$, and $\det q$ stands for the determinant of spatial metric q_{ab} . A and E take values in $\mathfrak{su}(2)$ algebra and $\mathfrak{su}(2)^*$ dual algebra respectively. In the case of symmetry reduced model (for the case $k = -1$ see (2.3)) the above equations simplify dramatically. (2.11) reduce to

$$A_a^i = -\varepsilon^{1i}{}_j \omega_a^j + \tilde{c} \omega_a^i \quad E_i^a = \tilde{p} \sqrt{\det q} e_i^a, \quad (2.12)$$

where $\tilde{c} = \gamma \dot{a}$ and $\tilde{p} = a^2$. Notice that connection A is not diagonal. This is very different situation than $k = 0$ and $k = 1$. One can check, that Gauss and diffeomorphism constraints in variables (2.12) are satisfied automatically. The only non-trivial one is scalar constraint. From (2.10) and (2.12) we get

$$H_{\text{gr}} = -\frac{3V_0}{8\pi G \gamma^2} \sqrt{|\tilde{p}|} (\tilde{c}^2 - \gamma^2), \quad (2.13)$$

where \tilde{c} and \tilde{p} are canonically conjugated $\{\tilde{c}, \tilde{p}\} = \frac{8\pi G \gamma}{3V_0}$ and $N(t) = 1$. V_0 is a volume of elementary cell (see [6, 7, 15] for details). In the presence of matter (in the isotropic and homogeneous case) the term $H_{\text{matt}} = V_0 |\tilde{p}|^{3/2} \rho_{\text{matt}}$ is added to gravitational scalar constraint

$$H^{\text{tot}} = -\frac{3V_0}{8\pi G \gamma^2} \sqrt{|\tilde{p}|} (\tilde{c}^2 - \gamma^2) + V_0 |\tilde{p}|^{3/2} \rho_{\text{matt}}. \quad (2.14)$$

If H^{tot} is constrained to vanish, one can easily check that the Friedmann equation (2.2) is recovered (for $k = -1$). We showed then followed by [19] that indeed the isotropic Bianchi V (class B) model has correct Hamiltonian formulation.

3 Quantum Theory

3.1 Kinematics

Quantum kinematics in the full Loop Quantum Gravity is based on the classical Poisson bracket algebra between holonomy along an edge $h_e[A]$ and fluxes (E smeared on 2-

surface) $E(S)$ [2, 18]. In the isotropic and homogeneous models $k = 0, 1$ holonomies are reduced to the so-called almost periodic functions $\sum_{\mu} \xi_{\mu} e^{\frac{i\mu c}{2}}$. Classical algebra $\{\tilde{p}, e^{\frac{i\mu \tilde{c}}{2}}\}$ is then easy to quantize and quantum theory is placed in the Bohr compactification of a real line [1, 4, 7, 15]. In the case of $k = -1$ the situation is more complicated [19]. Because the A connection is no longer diagonal the classical Poisson bracket algebra of the scale factor p with holonomies along the symmetry directions fails to be almost periodic functions, as well as holonomies. This means that one cannot construct the quantum algebra in the same Hilbert space (Bohr compactification of a real line). One of the possibilities is to abandon the A variable and use the connection γK_a^i (which for the $k = -1$ is diagonal) [19]. Then holonomies along left invariant vector fields ${}^o e_i^a \partial_a$ are in the form

$$h_i^{(\mu)} = \mathcal{P} \exp \left(\int_0^{\mu} ds \gamma K_a^k ({}^o e_i^a) \tau_k \right) = e^{\mu \tilde{c} \tau_i} = \mathbf{1} \cos \frac{\mu \tilde{c}}{2} + 2\tau_i \sin \frac{\mu \tilde{c}}{2}, \quad (3.1)$$

where μ is the length of an edge. Now it is easy to build quantum algebra of basic operators. Quantum version of the Classical Poisson bracket $\{p, e^{\frac{i\mu c}{2}}\} = -i\mu \frac{8\pi G}{6} e^{\frac{i\mu c}{2}}$ (after rescaling $\tilde{c} = V_0^{-1/3} c$ and $\tilde{p} = V_0^{-2/3} p$) is as follows

$$[\hat{p}, e^{\frac{i\mu c}{2}}] = \mu \frac{8\pi G \hbar \gamma}{6} e^{\frac{i\mu c}{2}}. \quad (3.2)$$

These operators act on vectors form the Hilbert space $\mathcal{H}^{\text{gr}} = L^2(\mathcal{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$. Eigenstates of \hat{p} consistute an orthonormal basis $\langle \mu' | \mu \rangle = \delta_{\mu' \mu}$ in \mathcal{H}^{gr}

$$\hat{p} |\mu\rangle = \mu \frac{8\pi l_{\text{Pl}}^2 \gamma}{6} |\mu\rangle, \quad (3.3)$$

where we denoted $G\hbar := l_{\text{Pl}}^2$. The spectrum of \hat{p} is then discrete. Each state from \mathcal{H}^{gr} can be decomposed in the $|\mu\rangle$ basis as a countable sum $|\psi\rangle = \sum_{\mu} \psi(\mu) |\mu\rangle$. The norm of $|\psi\rangle$ is then defined as

$$\langle \psi | \psi \rangle = \sum_{\mu} \bar{\psi}(\mu) \psi(\mu). \quad (3.4)$$

Using classical relation between the physical volume of the elementary cell and the scale factor $V = |p|^{3/2}$ one can easily construct the volume operator which is also diagonal in the $|\mu\rangle$ basis

$$\hat{V} |\mu\rangle = |\mu|^{3/2} \left(\frac{8\pi \gamma}{6} \right)^{3/2} l_{\text{Pl}}^3 |\mu\rangle. \quad (3.5)$$

The holonomy matrix element operator (3.1) acts as translations in $|\mu\rangle$ basis

$$e^{i \frac{\mu' c}{2}} |\mu\rangle = |\mu' + \mu\rangle. \quad (3.6)$$

3.2 The Loop — preparation

The formula for the scalar constraint (2.10) is simplified for symmetry reduced $k = -1$ model to

$$H_{gr} = -\frac{1}{16\pi G\gamma^2} \int d^3x N(t) \frac{E_i^a E_j^b}{\sqrt{|\det E|}} \varepsilon^{ij}{}_k (\Lambda_{ab}^k - \gamma^2 \Omega_{ab}^k), \quad (3.7)$$

where the part of the curvature proportional to the γ^2 does not have any dynamical degrees of freedom ($\Omega_{ab}^k = 2\partial_{[a}\Gamma_{b]}^k + \varepsilon_{ij}{}^k \Gamma_a^i \Gamma_b^j$, where Γ is defined in (2.12)). The curvature 2-form of the $A - \Gamma = \gamma K$ connection reads

$$\Lambda_{ab}^k = \partial_{[a}\gamma K_{b]}^k + \varepsilon_{ij}{}^k \gamma^2 K_a^i K_b^j = (-C^k{}_{ij}\tilde{c} + \tilde{c}^2 \varepsilon_{ij}{}^k) \omega_a^i \omega_b^j, \quad (3.8)$$

where C_{ij}^k are defined in (2.5). Naively we could introduce the loop such that its holonomy gives two components, one proportional to structure constants $C^k{}_{ij}$ of symmetry algebra and the second proportional to structure constants of $su(2)$ algebra $\varepsilon_{ij}{}^k$. However, putting equation (3.8) into scalar constraint (3.7) we find

$$E_i^a E_j^b \varepsilon^{ij}{}_k \Lambda_{ab}^k = E_i^a E_j^b \varepsilon^{ij}{}_k \varepsilon_{lm}{}^k \gamma^2 K_a^l K_b^m. \quad (3.9)$$

The term in curvature Λ proportional to $C^k{}_{ij}$ vanishes. The only term proportional to $\tilde{c}^2 \varepsilon_{ij}{}^k$ contributes to classical Hamiltonian which generates dynamics. Let us denote

$$\Lambda_{\text{eff}ab}^k = \varepsilon_{ij}{}^k \tilde{c}^2 \omega_a^i \omega_b^j. \quad (3.10)$$

Notice that in (3.8) there are only three possibilities for each values of indices i, j and k : $C^k{}_{ij} = 0$ and $\varepsilon_{ij}{}^k \neq 0$, $C^k{}_{ij} \neq 0$ and $\varepsilon_{ij}{}^k = 0$ or $C^k{}_{ij} = 0$ and $\varepsilon_{ij}{}^k = 0$. There are no such i, j, k for which $C^k{}_{ij}$ and $\varepsilon_{ij}{}^k$ contribute at the same “time”. This is very different situation to $k = +1$ model, where the terms proportional to \tilde{c} and \tilde{c}^2 contribute to the one and the same component of the curvature 2-form. Moreover, from (3.9) it is clear that the term proportional to $C^k{}_{ij}$ drops out in the scalar constraint. It is enough when we find the loop corresponding only to the \tilde{c}^2 term in the curvature (3.8), namely to the (3.10) .

3.3 The Loop

Let us now construct such a loop. We use technics developed in [7, 15]. The idea is to use the fact that left invariant vector commute with the right invariant ones. The left inv. fields are defined in (2.7) and the right inv. vector fields have the form [20]

$$\eta_1 = \partial_{x^1} - x^2 \partial_{x^2} - x^3 \partial_{x^3}, \quad \eta_2 = \partial_{x^2}, \quad \eta_3 = \partial_{x^3}. \quad (3.11)$$

It is easy to show, that $[e_i, \eta_j] = 0$ for every i and j . From the geometric interpretation of a Lie bracket of two vector fields it is clear that arbitrary pair of left and right invariant vector fields define a closed curve. Moreover, integral curves of those fields define in a natural way a surface spanned on the loop. In order to define coordinates on this surface we use a well known fact that every vector field on a given manifold generates one-parameter group of diffeomorphisms $\phi(t)$ which maps given point on a manifold \vec{x}_0 to $\vec{x}(t)$

$$\phi^{(t)} \vec{x}_0 = \vec{x}(t). \quad (3.12)$$

If given vector field has a form $V = f^a(x)\partial_a$ (where $f^a(x)$ are components in given coordinates) such a one-parameter group can be derived from the following condition

$$f^a(x) = \frac{dx^a}{dt}. \quad (3.13)$$

Lets us consider now an arbitrary point $\vec{x}_0 = (x_0^1, x_0^2, x_0^3)$ on Σ in the coordinate chart given by (2.7) and (3.11). One-parameter diffeomorphism generated by vector fields e_2 and η_3 can be written as

$$\begin{aligned} \phi_{(e_2)}^{(t)}(\vec{x}_0) &= (x_0^1, te^{-x_0^1} + x_0^2, x_0^3) \\ \phi_{(\eta_3)}^{(s)}(\vec{x}_0) &= (x_0^1, x_0^2, s + x_0^3). \end{aligned} \quad (3.14)$$

The holonomy (with respect to γK) along left invariant vector fields e_i is simple to calculate (3.1). What about the holonomy along right invariant fields? If we start from some point \vec{x}_0 on Σ , using the formula (2.7) and (3.11) we will find that the holonomy along η_3 has a form

$$h_{(\eta_3)} = \exp(se^{x_0^1} \tilde{c}\tau_3). \quad (3.15)$$

Such a holonomy depends on a starting point x_0^1 (notice, that the length of integral curve of η_3 with respect to the background metric $l_{(\eta_3)} = \int \sqrt{q_{ab}\eta_3^a\eta_3^b} = se^{x_0^1}$). Now, the loop is defined as follows: We start the holonomy around the loop from an arbitrary point \vec{x}_0 on Σ . Using 3.14 we get

1. From (x_0^1, x_0^2, x_0^3) we move along e_2 to the point $(x_0^1, te^{-x_0^1} + x_0^2, x_0^3)$
2. From $(x_0^1, te^{-x_0^1} + x_0^2, x_0^3)$ we move along η_3 to the point $(x_0^1, te^{-x_0^1} + x_0^2, s + x_0^3)$
3. From $(x_0^1, te^{-x_0^1} + x_0^2, s + x_0^3)$ we move to the point $(x_0^1, x_0^2, s + x_0^3)$ along e_2 but in the opposite direction than in 1)
4. From $(x_0^1, x_0^2, s + x_0^3)$ we move to the starting point (x_0^1, x_0^2, x_0^3) along η_3 , but in opposite direction than in 2).

What about the area of the surface spanned by oe_2 and ${}^o\eta_3$? The determinant of a metric tensor pulled back to the surface depends on the point of Σ

$$h := \det(h_{ab}) = e^{2x_0^1} \quad (3.16)$$

and the area (with respect to the background metric) is

$$\text{Ar} = \int dt \int ds \cdot \sqrt{h} = t s e^{x_0^1}. \quad (3.17)$$

If we take the length of an integral curve generated by oe_2 to be equal to μ , we can always choose such s in (3.17) and (3.15) that $s e^{x_0^1} = \mu$. Physical area of the surface can be constrained to be minimal $\text{Ar}_{\text{phy}} = \tilde{p} \bar{\mu}^2 = \Delta$ (see [6, 15] for details). Keeping this in mind and using (3.15), (3.1) and $\bar{\mu}$ condition we get a holonomy around the loop

$$h_{23}^{(\bar{\mu})} = e^{-\bar{\mu}c\tau_3} e^{-\bar{\mu}c\tau_2} e^{\bar{\mu}c\tau_3} e^{\bar{\mu}c\tau_2}. \quad (3.18)$$

As in [6, 15] shrinking the loop to zero we get the curvature 2-form

$${}^oe_2^a {}^oe_3^b \Lambda_{\text{eff}ab}^k = -2 \lim_{\bar{\mu} \rightarrow 0} \text{Tr} \frac{\tau_k h_{23}}{V_0^{2/3} \bar{\mu}^2} = \lim_{\bar{\mu} \rightarrow 0} \frac{\sin^2(\bar{\mu}c)}{V_0^{2/3} \bar{\mu}^2} \delta_1^k. \quad (3.19)$$

Because of homogeneity and isotropy ${}^oe_2^a {}^oe_3^b \Lambda_{\text{eff}ab}^k$ determines the $\Lambda_{\text{eff}ab}^k$ completely

$$\Lambda_{\text{eff}ab}^k = \lim_{\bar{\mu} \rightarrow 0} \frac{\sin^2 \bar{\mu}c}{V_0^{2/3} \bar{\mu}^2} \varepsilon_{ij}^k {}^o\omega_a^i {}^o\omega_b^j. \quad (3.20)$$

Notice that when we shrink our loop to a point $\bar{\mu} \rightarrow 0$ we recover the important part of curvature 2-form (3.10) and this is all we need. Since $\sin \bar{\mu}c$, as well as $\bar{\mu}^{-2}$, is a well defined operator in kinematical Hilbert space, the curvature (3.20) corresponds to well defined operator in $\mathcal{H}^{\text{gr}} = L^2(\mathcal{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$.

3.4 Quantum Dynamics

Using results from the previous section we can write classical scalar constraint regularized and rescaled by a factor of $16\pi G$

$$C_{\text{gr}}^{\text{reg}} = -\frac{6}{\gamma^2} \sqrt{|p|} \left(\frac{\sin^2 \bar{\mu}c}{\bar{\mu}^2} - V_0^{2/3} \gamma^2 \right), \quad (3.21)$$

where we have used the rescaled c and p variables. While the term

$$\sin \bar{\mu}c = \frac{1}{2i} (\exp(i\bar{\mu}c) - \exp(-i\bar{\mu}c)) \quad (3.22)$$

corresponds to the well defined operator in $\mathcal{H}^{\text{gr}} = L^2(\mathcal{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$ (i.e. translations in volume $\nu = K \text{sgn}(\mu)|\mu|^{3/2}$, see [6, 15] for details) the \sqrt{p} is quantized from the classical expression in the spirit of the full LQG in the following manner

$$\text{sgn}(p)\sqrt{|p|} = \frac{4}{3\kappa\gamma\mu} \sum_k \text{Tr} \left(h_k^{(\bar{\mu})} \{h_k^{(\bar{\mu})^{-1}}, V\} \tau_k \right). \quad (3.23)$$

When we put equations (3.21), (3.22) and (3.23) together we get \hat{C}_{gr} operator. Its action on state $|\psi\rangle = \sum_{\nu} \psi(\nu)|\nu\rangle$ is given by

$$\hat{C}_{\text{gr}}\psi(\nu) = f_{(+)}(\nu)\psi(\nu+4) + f_{(0)}(\nu)\psi(\nu) + f_{(-)}(\nu)\psi(\nu-4), \quad (3.24)$$

where the functions $f_{(\pm)}$ are defined as

$$\begin{aligned} f_{(+)}(\nu) &= \frac{27}{16} \sqrt{\frac{8\pi}{6\gamma^3}} K l_{\text{Pl}} |\nu+2| |\nu+1| - |\nu+3|, \\ f_{(-)}(\nu) &= f_{(+)}(\nu-4), \\ f_{(0)}(\nu) &= -f_{(+)}(\nu) - f_{(-)}(\nu) + A(\nu) \end{aligned} \quad (3.25)$$

and $A(\nu)$ is

$$A(\nu) = 3V_0^{2/3} \sqrt{\frac{8\pi\gamma}{6}} l_{\text{Pl}} \left(\frac{|\nu|}{K} \right)^{1/3} ||\nu+1| - |\nu-1||. \quad (3.26)$$

This way we have found the same scalar constraint operator as the one in [19]! It is then possible to interpret the Vandersloot [19] operator in the spirit of the full (LQG) theory: the curvature 2-form is replaced by the holonomy along a closed curve in the crucial scalar constraint operator. However, the holonomy used in the present paper and in [19] is considered as a function of γK rather than A variable.

4 Properties of the quantum scalar constraint operator – Universe with negative cosmological constant

The 3.24 operator defined in the previous section has the following properties

- It is densely defined in $\mathcal{H}^{\text{gr}} = L^2(\mathcal{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$ with the domain

$$\mathcal{D} = \left\{ |\psi\rangle \in \mathcal{H}^{\text{gr}} : |\psi\rangle = \sum_{i=1}^n a_i |\nu_i\rangle, a_i \in \mathbb{C}, n \in \mathbb{N} \right\}, \quad (4.1)$$

where $|\nu\rangle$ is volume eigenstate.

- The operator \hat{C}_{gr} preserves every subspace \mathcal{H}_ϵ of \mathcal{H}^{gr}

$$\mathcal{H}_\epsilon = \text{Span } |\epsilon + 4\mathbf{n}\rangle \in \mathcal{H}^{\text{gr}}, n \in \mathbb{N} \quad (4.2)$$

where ϵ is an arbitrary real number. We have then the following decomposition into orthogonal subspaces

$$\mathcal{H}^{\text{gr}} = \overline{\bigoplus_{\epsilon} \mathcal{H}_\epsilon}. \quad (4.3)$$

- \hat{C}_{gr} is symmetric with respect to scalar product

$$\langle \psi | \phi \rangle = \sum_{\nu} \bar{\psi}(\nu) \phi(\nu). \quad (4.4)$$

4.1 Negative cosmological constant

Classical expression for the cosmological constant has a form $C_\Lambda = 2\text{sgn}(\Lambda)|p|^{3/2}|\Lambda|$ (do not confuse Λ with curvature in (3.7)) and its contribution to scalar constraint is of the following form

$$C'_{\text{gr}} = -\frac{6}{\gamma^2} \sqrt{|p|} (c^2 - V_0^{2/3} \gamma^2) + 2\text{sgn}(\Lambda)|p|^{3/2}|\Lambda|. \quad (4.5)$$

Because the volume operator $\hat{V} = |\hat{p}|^{3/2}$ (3.5) is known, it is simple to define \hat{C}'_{gr} operator

$$\hat{C}'_{\text{gr}} \psi(\nu) = \hat{C}_{\text{gr}} \psi(\nu) + 2\text{sgn}(\Lambda)|\Lambda| \left(\frac{8\pi\gamma}{6} \right)^{3/2} l_{\text{Pl}}^3 \frac{|\nu|}{K} \psi(\nu), \quad (4.6)$$

where we used the spectrum of the volume operator in terms of ν (see [6] for details). Let us now fix $\text{sgn}(\Lambda) = -1$. For the negative cosmological constant the following theorem holds:

Theorem : The operator \hat{C}'_{gr} defined in the domain \mathcal{D} is essentially self-adjoint.

Proof : If we rewrite the \hat{C}'_{gr} in the following form

$$\hat{C}'_{\text{gr}} = \underline{\hat{C}} + \hat{C}_0, \quad (4.7)$$

where \hat{C}_0 is essentially self-adjoint, then in order to prove the theorem it is enough to show that

$$\|\underline{\hat{C}}\psi\|^2 \leq \|\hat{C}_0\psi\|^2 + \beta\|\psi\|^2 \quad (4.8)$$

for each $\psi \in \mathcal{D}$ and some constant β ([16] V.4.6). The action of (4.6) can be written as

$$\hat{C}'_{\text{gr}} \psi(\nu) = \underline{\hat{C}}\psi(\nu) + \hat{C}_0\psi(\nu), \quad (4.9)$$

where

$$\begin{aligned}\hat{\underline{C}}\psi(\nu) &= f_{(+)}(\nu)\psi(\nu+4) + f_{(-)}(\nu)\psi(\nu-4) \\ \hat{C}_0\psi(\nu) &= \left(-f_{(-)}(\nu) - f_{(+)}(\nu) + A(\nu) - 2|\Lambda| \left(\frac{8\pi\gamma}{6}\right)^{3/2} l_{\text{Pl}}^3 \frac{|\nu|}{K}\right)\psi(\nu).\end{aligned}\quad (4.10)$$

\hat{C}_0 is multiplication operator so it is obviously essentially self-adjoint. For the norm of $\hat{\underline{C}}$ operator the following inequality holds

$$\|\hat{\underline{C}}\psi\|^2 = \|(f_{(+)}U_4 + f_{(-)}U_{-4})\psi\|^2 \leq 2\langle\psi(f_{(+)}^2 + f_{(-)}^2)|\psi\rangle, \quad (4.11)$$

where $U_{\pm 4}$ is a unitary translation operator in ν representation defined by $\exp(\pm 2i\bar{\mu}c)$ (see [6, 15] for details). The (4.11) was derived from the inequality $\|u+w\|^2 \leq 2\|u\|^2 + 2\|w\|^2$. To conclude, the condition (4.8) is enough to show that C_0^2 (from 4.10) can be written as follows

$$C_0^2 = 2f_{(+)}^2 + 2f_{(-)}^2 + f_1 + f_0, \quad (4.12)$$

where $f_1 > 0$ is a function coming from square of (4.10) and $f_0 > 0$ is a bounded function which we can always add and it does not change the self-adjointness of \hat{C}'_{gr} .

5 Conclusions

In this paper we have found a nice analogue of square used in [6]. Because of the non-commuting character of left invariant fields \mathcal{e}_i in the hyperbolic $k = -1$ geometry, the loop was constructed using both left and right invariant fields as in [7, 15]. The important feature of this loop is very natural implementation of so-called $\bar{\mu}$ condition (i.e. the physical area of the loop is constrained to be minimal and equal to the quantum of area [3] which leads to improved dynamics [6]). Perhaps it seems surprising that our quantum loop leads to exactly the same operator as introduced by Vandersloot in [19]. This comes from the fact that the trace of holonomy around our closed curve (3.18) is precisely the same as the trace of holonomy around the curve generated by each pair $\mathcal{e}_i, \mathcal{e}_j$ for $i \neq j$ which is not closed as was pointed in [19] (page 8). Because our two scalar constraint operators are exactly the same, the correct semi-classical limit of the quantum theory numerically established in [19] is completely insensitive with respect to our results. Moreover, from the point of view of quantum theory there are no differences between Vandersloot model and ours. There are differences in initial concepts, but they lead to the same quantum theory. However, assumptions presented in this paper are more natural from the full theory point of view.

In section 4 we have found essentially self-adjoint operator corresponding to scalar constraint with negative cosmological constant, but what is the situation when $\Lambda = 0$?

What about more physical case of the positive cosmological constant? Unfortunately the theorem described in ([16] V.4.6) can not be applied to that case due to the fact that inequality (4.8) no longer holds for $\Lambda \geq 0$. Moreover, the similar problem arises in the $k = 0$ and $k = 1$ models with positive cosmological constant when one wants to apply the above theorem. We hope that future investigations give answer to the question about self-adjoint extensions of scalar constraint operators with $\Lambda > 0$.

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